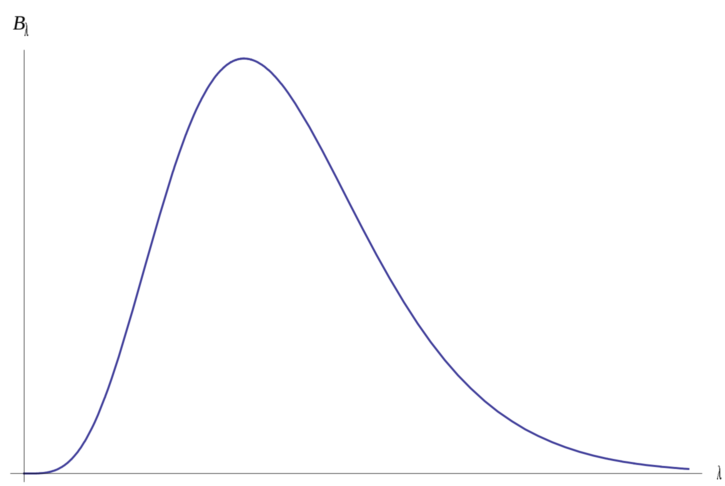
Teacher notes
Topic B

Planck's black body radiation law – last post in 2023.

A black body is a theoretical body that absorbs all the radiation incident on it reflecting none, a perfect absorber. To be in thermal equilibrium means that as much intensity is radiated as is absorbed and so a black body is also a perfect emitter, emitting the maximum possible intensity at a given temperature. The radiation emitted is electromagnetic radiation, i.e. photons with a range of wavelengths from zero to infinity. But not the same intensity is emitted at different wavelengths. Planck experimentally discovered that the intensity radiated per unit wavelength is given by

$$B_\lambda = \frac{2\pi hc^2}{\lambda^5}\frac{1}{e^{\frac{hc}{\lambda kT}}-1}$$

B_λ is called spectral intensity or spectral radiance. The intensity (in W m^{-2}) radiated in a wavelength interval dλ is $B_\lambda d\lambda$. Hence the total intensity radiated by the body is $I = \int_0^\infty B_\lambda \, d\lambda$ and is given by the area under the graph giving the variation of B_λ with λ:



To derive this law, Planck had to assume that the energy of the photons was a constant times the frequency. The constant is what we now call the Planck constant *h*.

It must be stressed that it is not correct to label the vertical axis of this graph as "intensity" or even "relative intensity". As we will see below, intensity is the total radiated power per unit area and is given by $I=\sigma T^4$ which shows no dependence on wavelength. Thus, a graph of intensity versus wavelength would be a horizontal straight line.

We can calculate the radiated intensity as follows:

Calling $x=\frac{hc}{\lambda kT}$ we get

$$B_\lambda=\frac{2\pi(kT)^5}{h^4c^3}\frac{x^5}{e^x-1}$$

The total intensity radiated is

$$I=\int_0^\infty B_\lambda\,d\lambda=\int_0^\infty\frac{2\pi(kT)^5}{h^4c^3}\frac{x^5}{e^x-1}\frac{d\lambda}{dx}dx$$

$$I=\int_\infty^0\frac{2\pi(kT)^5}{h^4c^3}\frac{x^5}{e^x-1}\frac{hc}{kT}(-\frac{1}{x^2})dx$$

$$I=\frac{2\pi(kT)^4}{h^3c^2}\int_0^\infty\frac{x^3}{e^x-1}dx \qquad (\int_0^\infty\frac{x^3}{e^x-1}dx=\frac{\pi^4}{15}\text{ is a standard integral-see below})$$

$$I=\frac{2\pi(kT)^4}{h^3c^2}\frac{\pi^4}{15}$$

$$I=\frac{2\pi^5k^4}{15h^3c^2}T^4$$

$$I=\sigma T^4$$

where the Stefan-Boltzmann constant is defined to be

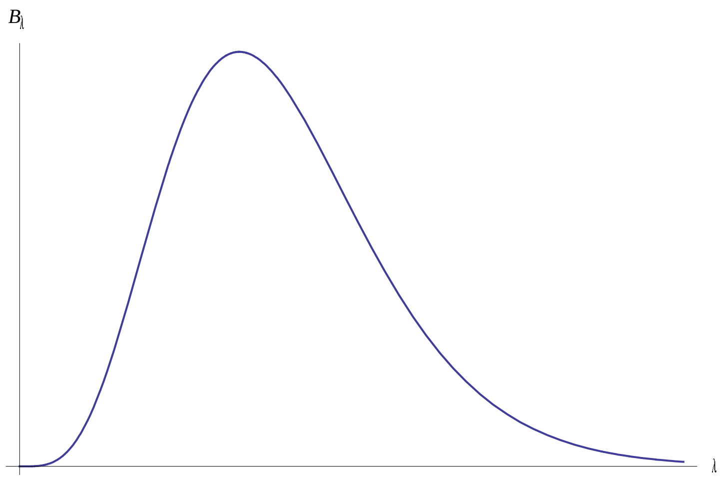
$$\sigma=\frac{2\pi^5k^4}{15h^3c^2}=\frac{2\pi^5\times(1.38\times10^{-23})^4}{15\times(6.63\times10^{-34})^3\times(2.998\times10^8)^2}=5.67\times10^{-8}\text{ W m}^{-2}\text{ K}^{-4}.$$

The radiated intensity $I=\sigma T^4$ is the area under the black body curve.

From the Planck law follows Wien's law which relates temperature to the wavelength at which the black body curve has its maximum: $\lambda_0 T=\text{constant}=2.9\times10^{-3}\text{ K m}$.

Questions

The black body curve for a body of area *A* and temperature *T* is given by the following graph.



(a) What happens to the graph if *A* is doubled?
(b) The temperature *T* of the body increases. How does the graph change?
(c) Which is hotter, a red star or a blue star?
(d) The Sun has a peak in its black body spectrum at a wavelength of about 500 nm. What is the surface temperature of the Sun?
(e) The wavelength in (d) is a blue-green wavelength. Is the color of the Sun blue-green then? Investigate this a bit on your own.

Derivation of the integral

$$\int_0^\infty \frac{x^3}{e^x - 1}\,dx = \int_0^\infty \frac{x^3 e^{-x}}{1 - e^{-x}}\,dx$$

$$\frac{1}{1 - e^{-x}} = \sum_{n=0}^{\infty} e^{-nx}$$

$$\int_0^\infty \frac{x^3}{e^x - 1}\,dx = \int_0^\infty \sum_{n=0}^{\infty} x^3 e^{-x} e^{-nx}\,dx = \int_0^\infty \sum_{n=0}^{\infty} x^3 e^{-(n+1)x}\,dx$$

$$= \sum_{n=0}^{\infty} \int_0^\infty x^3 e^{-(n+1)x}\,dx$$

Change variables to $t = (n+1)x \Rightarrow dt = (n+1)dx$ so that

$$\begin{aligned}
\int_0^\infty \frac{x^3}{e^x-1}\,dx &= \sum_{n=0}^{\infty}\int_0^\infty \frac{t^3e^{-t}}{(n+1)^4}\,dt \\
&= \sum_{n=0}^{\infty}\frac{1}{(n+1)^4}\int_0^\infty t^3e^{-t}\,dt \\
&= \sum_{n=0}^{\infty}\frac{1}{(n+1)^4}\Gamma(4) \\
&= \zeta(4)\Gamma(4) \\
&= \frac{\pi^4}{90}\times 3! \\
&= \frac{\pi^4}{15}
\end{aligned}$$

Answers to questions

(a) No change.

(b) The new curve is above the original curve and the peak is shifted to the left.

(c) Blue has shorter wavelength and by Wien's law a higher temperature.

(d) From $\lambda_0 T = \text{constant} = 2.9\times10^{-3}\ \text{K m}$, we get $T = \dfrac{2.9\times10^{-3}}{5.0\times10^{-7}} = 5800\ \text{K}$.